Math 4650 Homework 5 Solutions

$\boxed{1 (a)}$

Recall: f is continuous at a if for every $\varepsilon > 0$ there exists $\delta > 0$ where if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$

Let $a \in \mathbb{R}$.

Let $\varepsilon > 0$.

We want to find $\delta > 0$ where if $|x-a| < \delta$ then $|(2x+1) - (2a+1)| < \varepsilon$.

Note that

$$|(2x+1) - (2a+1)| = |2x - 2a| = |2||x-a| = 2|x-a|$$

Let $\delta = \frac{\varepsilon}{2}$.

Then if $|x-a| < \delta$, we get

$$|(2x+1) - (2a+1)| = 2|x-a| < 2 \cdot \delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon.$$

Thus, $f(x) = 2x+1$ is continuous at $a \in \mathbb{R}$.

$\blacksquare$

$\boxed{①(b)}$

Recall: f is continuous at a if for every $\varepsilon > 0$ there exists $\delta > 0$ where if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$

Let $a \in \mathbb{R}$.

Let $\varepsilon > 0$.

We want to find $\delta > 0$ where if $|x-a| < \delta$ then $|x^4 - a^4| < \varepsilon$

Note that

$$
\begin{aligned}
|x^4 - a^4| &= |(x^2+a^2)(x^2-a^2)| \\
&= |(x^2+a^2)(x+a)(x-a)| \\
&= \underbrace{|x^2+a^2|\,|x+a|}_{\text{let's get these bounded first}}\,\underbrace{|x-a|}_{\text{we can control this with } \delta}
\end{aligned}
$$

Suppose $0 < \delta < 1$ ← arbitrary starting bound I picked to bound $|x^2+a^2|$ and $|x+a|$

Suppose $|x-a| < \delta < 1$.

Then, $|x| = |x-a+a| \leq |x-a| + |a| < \delta + |a| < 1 + |a|$

So, $|x^2+a^2| \leq |x^2| + |a^2| = |x|^2 + |a|^2 < (1+|a|)^2 + |a|^2$

$= 1 + 2|a| + 2|a|^2$

And, $|x+a| \leq |x| + |a| < (1+|a|) + |a| = 1 + 2|a|$.

Thus, if $|x-a| < \delta < 1$, then

$$|x^4 - a^4| = |x^2 + a^2||x+a||x-a|$$
$$< (1 + 2|a| + 2|a|^2)(1+2|a|)|x-a|$$

Let $\delta < \min\left\{1, \dfrac{\varepsilon}{(1+2|a|+2|a|^2)(1+2|a|)}\right\}$.

Then if $|x-a| < \delta$ we get

$$|x^4 - a^4| < (1+2|a|+2|a|^2)(1+2|a|)|x-a|$$

since $\delta < 1$ from above

$$< (1+2|a|+2|a|^2)(1+2|a|) \cdot \frac{\varepsilon}{(1+2|a|+2|a|^2)(1+2|a|)}$$

$$= \varepsilon.$$

So, if $|x-a| < \delta$, then $|x^4 - a^4| < \varepsilon$.

Therefore $f(x) = x^4$ is continuous at all $a \in \mathbb{R}$. $\blacksquare$

$\boxed{①(c)}$

Recall: f is continuous at a if for every $\varepsilon > 0$ there exists $\delta > 0$ where if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$

Let $a \in \mathbb{R}$.

Let $\varepsilon > 0$.

We want to find $\delta > 0$ where if $|x-a| < \delta$ then $|(x^2+x) - (a^2+a)| < \varepsilon$.

Note that

$$
\begin{aligned}
|(x^2+x)-(a^2+a)| &= |(x^2-a^2)+(x-a)| \\
&\le |x^2-a^2| + |x-a| \\
&= |x+a||x-a| + |x-a| \\
&= \underbrace{(|x+a|+1)}_{\text{we will make an arbitrary starting bound on } \delta \text{ to bound this part}} \underbrace{|x-a|}_{\text{we can bound this with } \delta}
\end{aligned}
$$

Suppose $\delta < 1$.

Then if $|x-a| < \delta < 1$ we get that

$$|x| = |x-a+a| \le |x-a| + |a| < \delta + |a| < 1 + |a|$$

which gives

$$|x+a| \le |x| + |a| < (1+|a|) + |a| = 1 + 2|a|.$$

Thus, if $|x-a| < \delta < 1$ then

$$\begin{aligned} |(x^2+x)-(a^2+a)| &\le (|x+a|+1)\,|x-a| \\ &< ((1+2|a|)+1)\,|x-a| \\ &= (2+2|a|)\,|x-a| \end{aligned}$$

Let $\delta < \min\left\{1, \frac{\varepsilon}{2+2|a|}\right\}$

Then $\delta < 1$ and $\delta < \frac{\varepsilon}{2+2|a|}$.

Then if $|x-a| < \delta$ we get that

$$\begin{aligned} |(x^2+x)-(a^2+a)| &< (2+2|a|)\,|x-a| \\ &< (2+2|a|) \cdot \frac{\varepsilon}{2+2|a|} \\ &= \varepsilon \end{aligned}$$

since $\delta < 1$ from above

since $|x-a| < \delta < \frac{\varepsilon}{2+2|a|}$

Thus, if $|x-a| < \delta$, then $|(x^2+x)-(a^2+a)| < \varepsilon$.

So, $f(x) = x^2 + x$ is continuous at $a \in \mathbb{R}$ $\boxed{/\!/}$

$\boxed{1(d).}$

Recall: f is continuous at a if for every $\varepsilon > 0$ there exists $\delta > 0$ where if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$

Let $a > 0$.

Let $\varepsilon > 0$.

We want to find $\delta > 0$ where if $|x-a| < \delta$ then $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon$.

$x^2 > 0,\ a^2 > 0$

Note that if $x \neq 0$, then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \left|\frac{a^2 - x^2}{x^2 a^2}\right| = \frac{|x^2 - a^2|}{|x^2||a^2|} = \frac{|x+a||x-a|}{x^2 a^2}$$

Suppose that $0 < \delta < \frac{a}{2}$

starting bound on δ so we can bound the term $\frac{|x+a|}{x^2 a^2}$ in the above

Suppose that $|x-a| < \delta < \frac{a}{2}$.

Then, $-\frac{a}{2} < x - a < \frac{a}{2}$

so, $\frac{a}{2} < x < \frac{3a}{2}$.

see picture on right

Thus, $\frac{3a}{2} < x + a < \frac{5a}{2}$

So, $|x+a| = x + a < \frac{5a}{2}$

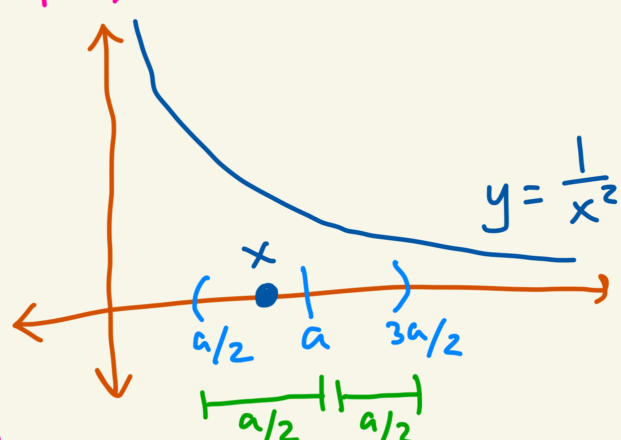
$x+a>0$ here

And, $\frac{a^2}{4} < x^2 < \frac{9a^2}{4}$

So, $\frac{1}{x^2} < \frac{4}{a^2}$

$\frac{a^2}{4} < x^2$

Note that $\frac{a}{2} > 0$ since we assumed $a > 0$. Also I picked this to keep us away from the asymptote at $x = 0$



Thus, if $|x-a| < \delta < \frac{a}{2}$, then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \frac{|x+a|}{x^2 a^2}|x-a|$$

$$= \frac{|x+a|}{a^2} \cdot \frac{1}{x^2} \cdot |x-a|$$

$$< \frac{(5a/2)}{a^2} \cdot \frac{4}{a^2}|x-a|$$

$$= \frac{10}{a^3}|x-a|$$

Let $\delta < \min\left\{1, \varepsilon/\left(\frac{10}{a^3}\right)\right\}$

Then, $\delta < 1$ and $\delta < \varepsilon/(10/a^3)$

Then if $|x-a| < \delta$ we get that

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \frac{10}{a^3}|x-a|$$

since $\delta < 1$ from above

$$< \frac{10}{a^3} \cdot \left(\varepsilon/(10/a^3)\right)$$

$$= \varepsilon.$$

Thus, if $|x-a| < \delta$, then $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon$.

So, $f(x) = \frac{1}{x^2}$ is continuous at $a > 0$. $\blacksquare$

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($\Rightarrow$) Suppose that f is continuous at a.

Let (x_n) be a sequence contained in D with $x_n \to a$.

Let $\varepsilon > 0$.

Since f is continuous at a there exists $\delta > 0$ where if $x \in D$ and $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

Since $x_n \to a$ there exists $N > 0$ where if $n \geq N$, then $|x_n - a| < \delta$.

Thus, if $n \geq N$, then $x_n \in D$ and $|x_n - a| < \delta$ and thus $|f(x_n) - f(a)| < \varepsilon$.

Hence $\lim\limits_{n \to \infty} f(x_n) = f(a)$

($\Leftarrow$) Suppose that given any sequence (x_n) contained in D with $x_n \to a$ we have that $\lim\limits_{n \to \infty} f(x_n) = f(a)$.

Let's use this to prove that f is continuous at a.

Suppose instead that f is not continuous at a.
Then there must exist a particular $\varepsilon_0 > 0$ where no matter what $\delta > 0$ is chosen there exists $\hat{x} \in D$ with $|\hat{x} - a| < \delta$ but $|f(\hat{x}) - f(a)| \geq \varepsilon_0$.

Consider $\delta_n = \frac{1}{n}$.

Then there exists a sequence (x_n) where for each n we have $x_n \in D$ and $|x_n - a| < \frac{1}{n}$ but $|f(x_n) - f(a)| \geq \varepsilon_0$

This gives us a sequence (x_n) contained in D with $x_n \to a$ but $\lim_{n\to\infty} f(x_n) \neq f(a)$

Contradiction. $\blacksquare$

3 Let f be continuous at $a \in \mathbb{R}$.

Suppose $f(a) > 0$.

Set $\varepsilon = f(a) > 0$.

Since f is continuous at a there exists $\delta > 0$ where if $|x-a| < \delta$ then $|f(x) - f(a)| < \underbrace{f(a)}_{\varepsilon}$.

Then if $|x-a| < \delta$ we get that

$$-f(a) < f(x) - f(a) < f(a).$$

So if $|x-a| < \delta$, then

$$0 < f(x) < 2f(a).$$

Thus, if $|x-a| < \delta$, then $0 < f(x)$.

$\square$

4 Let f be continuous on $\mathbb{R}$.

Let $S = \{x \mid f(x) = 0\} \neq \emptyset$.

Let (x_n) be a sequence of points from S such that $x_n \to L$.

Since (x_n) is contained in S we know $f(x_n) = 0$ for all n.

Let $\varepsilon > 0$.

Since f is continuous at every point in $\mathbb{R}$, f is continuous at L, thus there exists $\delta > 0$ where if $|x - L| < \delta$, then $|f(x) - f(L)| < \varepsilon$.

Since $x_n \to L$ there exists $N > 0$ where if $n \geq N$ then $|x_n - L| < \delta$.

Thus, if $n \geq N$ then $|f(x_n) - f(L)| < \varepsilon$

So, if $n \geq N$ then $|0 - f(L)| < \varepsilon$ $\leftarrow$ $f(x_n) = 0$ for every n

Thus, $|f(L)| < \varepsilon$.

Since $|f(L)| < \varepsilon$ for any positive ε we must have $|f(L)| = 0$

Thus, $f(L) = 0$.

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Let $\varepsilon > 0$.

Since $L \in D$, f is continuous at L.

Thus there exists $\delta_1 > 0$ where if $y \in D$ and $|y - L| < \delta_1$, then $|f(y) - f(L)| < \varepsilon$

Since $\lim_{x \to a} g(x) = L$, there exists $\delta_2 > 0$ where if $x \in A$ and $0 < |x-a| < \delta_2$ then $|g(x) - L| < \delta_1$.

(Note by assumption if $x \in A$ then $g(x) \in D$.)

From above, if $x \in A$ and $0 < |x-a| < \delta_2$, then $|f(g(x)) - f(L)| < \varepsilon$.

Thus, $\lim_{x \to a} f(g(x)) = f(L)$.

Note: In the above we were showing that $\lim_{x \to a} (f \circ g)(x) = f(L)$ and the domain of $f \circ g$ here is A. Hence the "$x \in A$".

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(a) Let $d \in \mathbb{R}$.
Let $\varepsilon > 0$.
Set $\delta = \varepsilon$.
Then, if $|x-d| < \delta$,
we have $|f(x) - f(d)| = |x-d| < \delta = \varepsilon$.
So, $f(x) = x$ is continuous at d.

(b) Let $a \in \mathbb{R}$.
Let $\varepsilon > 0$.
Set $\delta = 1$. ← you can pick any positive number
If $|x-d| < 1$, then $|f(x) - f(d)| = |\alpha - \alpha| = 0 < \varepsilon$.
Thus, f is continuous at d.

(c) Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$
be a polynomial where $a_n, a_{n-1}, \dots, a_1, a_0 \in \mathbb{R}$.
Let $d \in \mathbb{R}$.
By part (a) we know x is continuous at d.
From class if we product two functions
that are both continuous at d then
the resulting function is continuous at d.
Thus, $x, x^2, x^3, x^4, \dots, x^n$ are continuous at d.

Since $a_1, a_2, \ldots, a_n$ are continuous at d, by part (b) of this problem, by the same reasoning we have that $a_1 x, a_2 x^2, a_3 x^3, \ldots, a_n x^n$ are all continuous at d.

From class if we add two functions that are both continuous at d then the resulting function is continuous at d.

Since $a_0, a_1 x, a_2 x^2, \ldots, a_n x^n$ are all continuous at d we get that $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is continuous at d.